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1/4-BPS states on noncommutative tori

Anatoly Konechny and Albert Schwarz

Department of Mathematics, University of California
Davis, CA 95616 USA
E-mail: konechny@math.ucdavis.edu, schwarz@math.ucdavis.edu

Abstract: We give an explicit expression for classical 1/4-BPS fields in supersymmetric Yang-Mills theory on noncommutative tori. We use it to study quantum 1/4-BPS states. In particular we calculate the degeneracy of 1/4-BPS energy levels.

Keywords: M(atrix) Theories, M-Theory, String Duality.

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Contents

1. Introduction

Let θ be an antisymmetric bilinear form on a lattice $\mathbb{Z}^d$. An algebra T_θ of functions on a noncommutative torus can be defined as an algebra with generators $U_{\mathbf{k}}$ labelled by lattice vectors $\mathbf{k} \in \mathbb{Z}^d$ and satisfying the relations

$$U_{\mathbf{k}_1} U_{\mathbf{k}_2} = \exp(\pi i \langle \mathbf{k}_1 \theta \mathbf{k}_2 \rangle) U_{\mathbf{k}_1 + \mathbf{k}_2} \,. \tag{1.1}$$

Consider derivations δ_i of T_θ defined by the following relations

$$\delta_j U_{\mathbf{k}} = 2\pi i k_j \; U_{\mathbf{k}} \,.$$

This derivations span an abelian Lie algebra which we denote by L_θ. Given a homomorphism of a Lie algebra L into L_θ we define a connection with respect to L as follows. A connection ∇_X on a T_θ module E is a set of linear operators $\nabla_X : E \to E$, $X \in L$ satisfying the Leibnitz rule

$$\nabla_X(ae) = a \nabla_X e + (\delta_X a) e$$

for any $a \in T_\theta$, $e \in E$. Here δ_X stands for the image of $X \in L$ in L_θ under the homomorphism introduced above. We also assume that δ_X and ∇_X depend linearly on X. Below we are interested in the case when L is a ten-dimensional abelian Lie algebra endowed with a metric. In addition we fix a basis in L consisting of vectors X_0, X_i, $i = 1, \ldots, d$, X_I, $I = d+1, \ldots, 9$ such that $\delta_{X_i} = \delta_i$, $\delta_I = 0$ and the metric tensor $g_{\mu\nu}$ in this basis satisfies $g_{00} = -1$, $g_{i0} = g_{I0} = 0$, $g_{iJ} = 0$, $g_{IJ} = \delta_{IJ}$. (Note that the index conventions in the present paper are different from the ones used in our preceding paper [7].)

It is shown in [3] that toroidal compactification of matrix model ([1, 2]) leads to Yang-Mills theory on a noncommutative torus. More precisely, the Minkowski action functional of compactified theory takes the form

$$S = \frac{-V}{4g_{YM}^2} \operatorname{Tr}(F_{\mu\nu} + \phi_{\mu\nu} \cdot \mathbf{1})(F^{\mu\nu} + \phi^{\mu\nu} \cdot \mathbf{1}) + \frac{iV}{2g_{YM}^2} \operatorname{Tr} \bar{\psi} \Gamma^I [\nabla_I, \psi] \,.$$

Here $F_{\mu\nu}$ is the curvature of connection ∇_μ defined with respect to an abelian ten-dimensional Lie algebra L on a projective module E over T_θ , $\phi_{\mu\nu}$ is a constant antisymmetric tensor whose only non-vanishing components are ϕ_{ij}, $i, j = 1, \ldots, d$, ψ is a ten dimensional Majorana-Weyl spinor taking values in the algebra of endomorphisms $\mathrm{End}_{T_\theta} E$. This action functional is invariant under the following supersymmetry transformations

$$\begin{aligned} \delta_{\epsilon,\tilde{\epsilon}} \nabla_I &= \frac{i}{2} \bar{\epsilon} \Gamma_I \psi \\ \delta_{\epsilon,\tilde{\epsilon}} \psi &= -\frac{1}{4} F_{IJ} \Gamma^{IJ} \epsilon + \tilde{\epsilon} \cdot \mathbf{1} \, . \end{aligned} \tag{1.2}$$

It is clear from (1.2) that constant curvature connections on T_θ can be identified with 1/2-BPS fields (see [3] for details). In papers [6, 7] we studied constant curvature connections and quantum states that arise by quantization of these fields and their fluctuations. Using the constructions of [6] and [7] one can obtain not only the energies of the 1/2-BPS states but also the energies of other quantum states, in particular, 1/4-BPS states. In the present paper we give a direct construction of 1/4-BPS classical fields and use it to study quantum 1/4-BPS states. We restrict ourselves to the case when the matrix θ_{ij} is irrational and, moreover, any linear combination of its entries with integer coefficients is irrational. Most of our results do not depend on this restriction. However, it permits us to simplify drastically our considerations. Namely, using the above restriction we will be able to reduce the problems we consider to the case of a free module (the notion of Morita equivalence and the results of [4, 5, 6] play a crucial role in this reduction).

Let us formulate our results first for the case $d = 2$. A projective module E over a two-dimensional noncommutative torus T_θ can be characterized by means of two integers p and q obeying $p - q\theta > 0$. Let us also assume that p and q are relatively prime. A 1/4-BPS state on E is characterized by topological numbers p, q and integers m_i, n^j, $i, j = 1, 2$ that specify the eigenvalues of operators $\mathcal{P}_i = \mathrm{Tr}\, F_{ij} P^j$ and $p^i = \mathrm{Tr}\, P^i$. (Here P^i stands for the momentum canonically conjugated to ∇_i in the hamiltonian formalism).

We will obtain the following expression for 1/4-BPS energy spectrum

$$\begin{aligned} E_{p,q,m_i,n^j} = & \frac{g_{YM}^2}{2V \dim E} (n^i + \theta^{ik} m_k) g_{ij} (n^j + \theta^{jl} m_l) + \\ & + \frac{1}{2V g_{YM}^2 \dim E} (\pi q + \phi_{12} \dim E)^2 + \frac{2\pi}{\dim E} \sqrt{K_i g^{ij} K_j} \end{aligned} \tag{1.3}$$

where $K_i = p m_i - q_{ij} n^j$. The degeneracy of E_{p,q,m_i,n^j} is given by the number $c(K)$ where $K = \gcd(K_i)$ and $c(K)$ is the coefficient at x^K in the Taylor expansion of the function

$$Z(x) = 2^8 \prod_n \left(\frac{1 + x^n}{1 - x^n} \right)^8 . \tag{1.4}$$

For $d=3$ projective modules are labelled by an integer p and an antisymmetric 3×3 matrix q_{ij} with integer entries. These numbers have to satisfy $\dim E = p + \frac{1}{2}\operatorname{tr}\theta q > 0$. The 1/4-BPS spectrum reads

$$\begin{aligned} E = {} & \frac{g_{YM}^2}{2V\dim E}(n^i+\theta^{ik}m_k)g_{ij}(n^j+\theta^{jl}m_l)+\\ & +\frac{V}{4\dim E g_{YM}^2}(\pi q_{ij}+\dim E\phi_{ij})g^{ik}g^{jl}(\pi q_{kl}+\dim E\phi_{kl})+\\ & +\pi\sqrt{v_i g^{ij} v_j} \end{aligned} \tag{1.5}$$

where $v_i = m_i\dim E - q_{ij}(n^j+\theta^{jk}m_k)$. The degeneracy of this eigenvalue is given by $c(K)$ obtained from (1.4) for $K=\gcd(pm_i - q_{ij}n^j, m_i\frac{1}{2}\epsilon^{ijk}q_{jk})$. This expression agrees with the expression obtained in [13] for the commutative case. We do not consider the case when (p,q_{ij}) are not relatively prime. The results for this case can be derived from our considerations combined with the results of [15, 16]. The eigenvalues (1.3), (1.5) coincide with the BPS bounds obtained in [7] from supersymmetry algebra. We work in the hamiltonian formalism on a three-dimensional noncommutative torus. Instead of that we could consider 1/4-BPS states using the euclidean lagrangean formalism on a four-dimensional noncommutative torus. In this setting the problem is related to the study of cohomology of the moduli space of noncommutative instantons. This means that one can obtain information about these cohomology groups from our calculations.

For $d>3$ we are able to analyze the degeneracy of 1/4-BPS states only in modules admitting a constant curvature connection. (For $d=2,3$ every module has this property.) It was shown in [6, 8] that such modules can be characterized by the property that corresponding K-theory class μ is a generalized quadratic exponent. We impose also an additional requirement that the module cannot be represented as a direct sum of equivalent modules (i.e. μ is not divisible by integer larger than 1 in the K-group). Such modules were called basic modules in [6]. We calculate the spectrum of 1/4-BPS states and the corresponding multiplicities in basic modules. The multiplicity again is given by a number $c(K)$ where the expression for K in terms of topological numbers is given in section 3.

2. Classical 1/4 BPS solutions

We start with a brief discussion of the hamiltonian formalism in the model at hand. In [6] we discussed the quantization of Yang-Mills theory on a noncommutative torus in the $\nabla_0=\partial_t$ gauge. Those considerations can be easily generalized to the supersymmetric case. Let us briefly describe the quantization procedure. The Minkowski action functional is defined on the configuration space $\operatorname{Conn} E\times(\Pi\operatorname{End}_{T_\theta}E)^{16}$ where $\operatorname{Conn} E$ denotes the space of connections on E, Π denotes the parity reversion operator. To describe the hamiltonian formulation we first restrict ourselves to the space

$\mathcal{M} = \mathrm{Conn}' E \times (\mathrm{End}_{T_\theta} E)^9 \times (\Pi\, \mathrm{End}_{T_\theta} E)^{16}$ where $\mathrm{Conn}' E$ stands for the space of connections satisfying $\nabla_0 = 0$, the second factor corresponds to a cotangent space to $\mathrm{Conn}' E$. We denote coordinates on that cotangent space by P^I. Let $\mathcal{N} \subset \mathcal{M}$ be a subspace where the gaussian constraint $[\nabla_i, P^i] = i\bar{\psi}\Gamma^0\psi$ is satisfied. Then the phase space of the theory is the quotient $\mathcal{P} = \mathcal{N}/G$ where G is the group of spatial gauge transformations.

The presymplectic form (i.e. a degenerate closed 2-form) ω on $\mathcal{M}$ is defined as

$$\omega = \mathrm{Tr}\, \delta P^I \wedge \delta \nabla_I - i\frac{V}{2g_{YM}^2}\, \mathrm{Tr}\, \delta\bar{\psi}\Gamma^0 \wedge \delta\psi\,. \tag{2.1}$$

It descends to a symplectic form on the phase space $\mathcal{P}$. The hamiltonian of the theory reads

$$\begin{aligned} H = {} & \mathrm{Tr}\, \frac{g_{YM}^2 R_0^2}{2V} P^I g_{IJ} P^J + \\ & + \mathrm{Tr}\, \frac{V}{4g_{YM}^2}(F_{IJ} + \phi_{IJ}\cdot \mathbf{1}) g^{IK} g^{JL} (F_{KL} + \phi_{KL}\cdot\mathbf{1}) + \\ & + \text{fermionic terms}\,. \end{aligned} \tag{2.2}$$

Let θ be an antisymmetric $d\times d$ matrix. We assume that any linear combination of its entries with integer coefficients is irrational. Consider the corresponding d-dimensional noncommutative torus T_θ. Let E be a projective module over it. In this paper we consider a particular class of modules introduced in [6] which we named basic modules. A module E is called basic if the algebra $\mathrm{End}_{T_\theta} E$ is a noncommutative torus $T_{\tilde\theta}$ and there is a constant curvature connection ∇_i on E that satisfies the condition $[\nabla_i, \phi] = \tilde\delta_i\phi$ for every $\phi \in \mathrm{End}_{T_\theta} E$ (here $\tilde\delta_i, i = 1, \ldots, d$ is a basis of the algebra $L_{\tilde\theta}$ of derivations of $T_{\tilde\theta}$). The condition that E is basic is equivalent to the condition that T_θ is completely Morita equivalent to the torus $T_{\tilde\theta} = \mathrm{End}_{T_\theta} E$ so that the module $\tilde E$ over $T_{\tilde\theta}$ corresponding to E under this equivalence is a one dimensional free module. Therefore, provided components of θ satisfy the irrationality condition above, θ and $\tilde\theta$ are related by an $\mathrm{SO}(d, d|\mathbb{Z})$ transformation

$$\tilde\theta = (M\theta + N)(R\theta + S)^{-1}\,, \tag{2.3}$$

where

$$g = \begin{pmatrix} M & N \\ R & S \end{pmatrix} \in \mathrm{SO}(d, d|\mathbb{Z})\,. \tag{2.4}$$

As specified in the definition a basic module E is equipped with a constant curvature connection. All modules over noncommutative tori admitting a constant curvature connection are classified in [8] (see also [6, appendix D]) in terms of their representatives in the K-group $K_0(E)$. An element $\mu(E) \in K_0(T_\theta)$ that corresponds to a module admitting a constant curvature connection has the form of a generalized quadratic exponent.

If T_θ is a torus of dimension $d = 2$ or $d = 3$, then an element in $K_0(E)$ representing a module E over T_θ is always a generalized quadratic exponent; it has the form

$$\mu(E) = p + \frac{1}{2}\alpha^i q_{ij}\alpha^j ,$$

where (p, q_{ij}) are integers. The dimension of such a module is given by the formula

$$\dim E = p + \frac{1}{2}\,\mathrm{tr}(\theta q) .$$

It follows from the results of [4] that under an $\mathrm{SO}(d, d|\mathbb{Z})$ transformation (2.3), (2.4) the numbers (p, q_{ij}) transform according to a spinor representation. The condition that $\mu(E)$ given above corresponds to a basic module is $\gcd(p, q_{ij}) = 1$.

Let us fix a basic module E over a d-dimensional torus T_θ and thus a generalized quadratic exponent $\mu = \mu(E)$. Denote by $Z_{\mathbf{k}}$, $\mathbf{k} \in \Gamma$ generators of the torus $T_{\tilde\theta} = \mathrm{End}_{T_\theta} E$. They satisfy the relations

$$Z_{\mathbf{k}} Z_{\mathbf{k}'} = e^{\pi i k_i \tilde\theta^{ij} k'_j} Z_{\mathbf{k}+\mathbf{k}'} . \tag{2.5}$$

Using the supersymmetry algebra of the model at hand (see [7]) one can describe all classical solutions preserving 1/4 of all supersymmetries which we call 1/4 BPS fields. Our discussion of these solutions here essentially parallels the one in [13] for the commutative case.

The equations defining 1/4 BPS fields are of the following form

$$\begin{aligned}
\psi &= 0\,, & [X_I, X_J] &= 0\,, & \\
[\nabla_i, \nabla_j] &= c_{ij}\cdot \mathbf{1}\,, & [\nabla_i, X_I] &= 0 & i,j = 1,\ldots,d-1\,,\\
P^d &= p^d \cdot \mathbf{1}\,, & P_I &= V g_{YM}^{-2} R_d^{-1} [\nabla_d, X_I]\,, &\\
& & P_i = c_i \cdot \mathbf{1} &+ V g_{YM}^{-2} R_d^{-1} [\nabla_d, \nabla_i]\,, &\\
& & \sum_{i=1}^{d-1} [\nabla_i, P^i] &+ \sum_{I=d+1}^{9} [X_I, P^I] = 0\,, &
\end{aligned} \tag{2.6}$$

where $R_d^{-1} = \sqrt{g^{dd}}$, and c_{ij}, c_i, p^d are constants. More precisely, equations (2.6) define 1/4-BPS fields that are invariant under the supersymmetry transformations (1.2) with ϵ, $\tilde\epsilon$ satisfying

$$\begin{aligned}
(\Gamma^0 + \Gamma^d)\epsilon &= 0\,,\\
\tilde\epsilon &= -\frac{1}{4}(c_{ij}\Gamma^{ij} + 2\Gamma^0\Gamma^i c_i + 2\Gamma^0\Gamma_d p^d)\epsilon\,.
\end{aligned} \tag{2.7}$$

In equations (2.6), and (2.7) we single out the d-th direction. In general a set of 1/4-BPS fields preserved by the same subgroup of supersymmetries determines a primitive lattice vector $\mathbf{m} \in \mathbb{Z}^d$. (The equations (2.6) are obtained if one changes a basis in the lattice $\mathbb{Z}^d$ so that $\mathbf{m}$ is the d-th basis vector.)

The last equation in (2.6) is the Gauss constraint. The gauge equivalence classes of solutions to (2.6) define a subspace in the phase space of the theory - a moduli space of 1/4 BPS fields. The presymplectic form (2.1) gives rise to a symplectic form on the moduli space and thus one can perform a quantization of the resulting theory. As any basic module (any module in the $d = 2, 3$ case) can be transformed by means of Morita equivalence to a free module we can restrict ourselves to consideration of a free module where the moduli space of 1/4 BPS fields can be most easily studied. Thus from now on we assume that $\tilde{E}$ is a rank 1 free module over the torus $T_{\tilde{\theta}} = \mathrm{End}_{T_\theta} E$.

It is easy to check that the following formulas define 1/4 BPS fields

$$
\begin{aligned}
\nabla_i &= \tilde{\delta}_i + \sum_{k\in\mathbb{Z}} A_i(k) Z_{k\mathbf{m}}, \qquad i = 1, \ldots, d-1 \\
P^j &= \tilde{p}^j \cdot \mathbf{1} + \tilde{V} g_{YM}^{-2} \tilde{R}_d^{-1} \left(\sum_{k\in\mathbb{Z}} ikA^j(k) Z_{k\mathbf{m}} \right) \qquad i = 1, \ldots, d-1 \\
P^d &= \tilde{p}^d \cdot \mathbf{1}\,, \qquad A_d = q_d \cdot \mathbf{1}\,, \qquad \psi = 0 \\
X_I &= \sum_{k\in\mathbb{Z}-\{0\}} X_I(k) Z_{k\mathbf{m}} \\
P^I &= \tilde{V} g_{YM}^{-2} \tilde{R}_d^{-1} \left(\sum_{k\in\mathbb{Z}} ikX^I(k) Z_{k\mathbf{m}} \right),
\end{aligned} \tag{2.8}
$$

where $\tilde{\delta}_i$ are derivations of $T_{\tilde{\theta}}$ acting according to the formula

$$
\tilde{\delta}_j Z_{\mathbf{k}} = 2\pi i k_j Z_{\mathbf{k}}\,. \tag{2.9}
$$

The last formula fixes a basis in the Lie algebra L, and $\tilde{V}$, $\tilde{R}_d$ refer to the metric tensor $\tilde{g}_{ij}$ in this basis. It is straightforward to check that the set of fields (2.8) also satisfies the Gauss constraint. Moreover, any solution to (2.6) can be brought to the form (2.8) by means of gauge transformations. Let us sketch a proof of this fact. Any connection on $\tilde{E}$ has the form $\nabla_\alpha = \tilde{\delta}_\alpha + A_\alpha$, $(\alpha = 1, \ldots, d)$ where A_α are endomorphisms of $\tilde{E}$, which are just elements of the torus $T_{\tilde{\theta}}$ acting on $\tilde{E}$ from the right (by a slight abuse of notation we denote the generators of endomorphisms of $\tilde{E}$ by the same letters as generators of the torus $T_{\tilde{\theta}}$). First, because of the flatness condition $[\nabla_i, \nabla_j] = 0$, $i, j = 1, \ldots, d-1$ we can bring the connection components ∇_i to the form

$$
\nabla_i = \tilde{\delta}_i + \sum_{k\in\mathbb{Z}} A_i(k) Z_{k\mathbf{m}}\,.
$$

Due to the equation $P_i = \tilde{V} g_{YM}^{-2} \tilde{R}_d^{-1} [\nabla_d, \nabla_i]$ the Gauss constraint $[\nabla_i, P^i] + [X_I, P^I] = 0$ now takes the form

$$\nabla_i \nabla^i (A_d) + [X_I, [X^I, A_d]] = 0 \,.$$

This equation implies that A_d is of the form

$$A_d = \sum_{k \in \mathbb{Z}} A_d(k) Z_{k\mathbf{m}} \,.$$

Finally, using gauge transformations generated by $Z_{k\mathbf{m}}$ we can bring A_d to the desired form $A_d = \mathrm{const} \cdot \mathbf{1}$.

Note that the space of fields having the form (2.8) is still invariant under a subgroup of gauge transformations that consists of transformations defined by monomials $Z_{\mathbf{k}}$. We denote this group by G^{mon}. The symplectic form (2.1) being restricted to the subspace (2.8) reads

$$\omega_{1/4} = \tilde{V} g_{YM}^{-2} \tilde{R}_d^{-1} \left(\sum_{k \in \mathbb{Z} - \{0\}} ik \left(\sum_{j=1}^{d-1} \delta A^j(k) \wedge \delta A_j(-k) + \right.\right.$$
$$\left.\left. + \sum_{I=d+1}^{9} \delta X_I(k) \wedge \delta X^I(-k) \right) \right) + \sum_{s=1}^{d} \delta \tilde{p}^s \wedge \delta q_s \,, \tag{2.10}$$

where q_s stands for the zero mode component $A_s(0)$, $s = 1, \ldots, d$. The hamiltonian has the form

$$H_{1/4} = \frac{g_{YM}^2}{2\tilde{V}} \sum_{s,r=1}^{d} \tilde{p}^s \tilde{g}_{sr} \tilde{p}^r + \frac{\tilde{V}}{4 g_{YM}^2} \phi_{\mu\nu} \phi^{\mu\nu} +$$
$$+ \tilde{V} g_{YM}^{-2} \tilde{R}_d^{-2} \left(\sum_{k \in \mathbb{Z} - \{0\}} k^2 \left(\sum_{i=1}^{d-1} A_i(k) A^i(-k) + \sum_{I=d+1}^{9} X_I(k) X^I(-k) \right) \right) . \tag{2.11}$$

The space of fields of the form (2.8) can be extended by adding fermionic degrees of freedom to a minimal supermanifold $\mathcal{B}_{1/4}$ invariant under all supersymmetry transformations. This is achieved by adding spinor fields $\psi = \sum_{k \in Z} \psi(k) Z_{k\mathbf{m}}$. The corresponding additional term to the hamiltonian (2.11) reads

$$H_{1/4}^{ferm} = \frac{V}{2 g_{YM}^2} \sum_{k \in \mathbb{Z}} ik \psi(k)^t \Gamma^d \psi(k) \,. \tag{2.12}$$

To describe 1/4-BPS states we should quantize the systems described by the hamiltonian $H_{1/4} + H_{1/4}^{ferm}$ on a symplectic manifold $\mathcal{B}_{1/4}/G^{\mathrm{mon}}$. (We can obtain the symplectic form on this manifold restricting the form (2.1) or adding fermionic terms to (2.10).) The combined system (2.11), (2.12) describes a free motion for the zero modes degrees of freedom q_s, $\psi(0)$, and an infinite system of supersymmetric harmonic oscillators with frequences $\omega(k) = \|k\| = \sqrt{k\tilde{g}^{dd}k}$. This system is a direct analogue of the chiral sigma model on $\mathbb{R}^6 \times T^2$ considered in [13, 14]. More precisely, we can introduce periodic functions

$$A_i(\phi) = \sum_{k\in\mathbb{Z}} A_i(k) e^{ik\phi} \qquad X_I(\phi) = \sum_{k\in\mathbb{Z}} X_I(k) e^{ik\phi} .$$

and express the hamiltonian and symplectic form in terms of these functions:

$$\begin{aligned} H &= \frac{g_{YM}^2}{2\tilde{V}} \sum_{s,r=1}^{d} \tilde{p}^s \tilde{g}_{sr} \tilde{p}^r + \frac{\tilde{V}}{4g_{YM}^2} \phi_{\mu\nu}\phi^{\mu\nu} + \\ &+ \tilde{V} g_{YM}^{-2} \tilde{R}_d^{-2} \int_0^{2\pi} d\phi \left(\sum_{i=1}^{d-1} \frac{dA^i}{d\phi}\frac{dA_i}{d\phi} + \sum_{I=d+1}^{9} \frac{dX^I}{d\phi}\frac{dX_I}{d\phi} \right), \end{aligned} \tag{2.13}$$

$$\begin{aligned} \omega &= \tilde{V} g_{YM}^{-2} \tilde{R}_d^{-1} \int_0^{2\pi} d\phi \left(\sum_{j=1}^{d-1} \frac{d(\delta A^j)}{d\phi} \wedge \delta' A_j + \sum_{I=d+1}^{9} \frac{d(\delta X^I)}{d\phi} \wedge \delta' X_I \right) + \\ &+ \sum_{s=1}^{d} \delta\tilde{p}^s \wedge \delta q_s . \end{aligned} \tag{2.14}$$

We obtained the hamiltonian and symplectic form of the standard chiral sigma-model. (We write down only the bosonic part of the model. Its supersymmetrization is straightforward.) However the phase space is not standard. We should factorize with respect to the action of group G^{mon}. In other words we identify $A_j(\phi)$ with $A_j(\phi + 2\pi\tilde{\theta}^{ds} n_s) + 2\pi i n_j$ and $X_I(\phi)$ with $X_I(\phi + 2\pi\tilde{\theta}^{ds} n_s)$ for any integer valued vector $(n_s) \in \mathbb{Z}^d$ that specifies an element of G^{mon}. For $\theta = 0$ this means that we consider the classical sigma-model on $T^{d-1} \times \mathbb{R}^{9-d}$; we will use this terminology also in the case $\theta \neq 0$ although one should emphasize that our sigma-model is not completely standard.

3. Quantization

For now let us concentrate on the bosonic part (2.11) of the model. Upon the hamiltonian quantization of the model (2.11), (2.10) we get a Hilbert space spanned by the wave functions

$$\Psi_{\tilde{p}_i; N_1(k),\ldots,N_8(k)} = \exp\left(i \sum_{s=1}^{d} \tilde{p}^s q_s \right) \prod_{k\in\mathbf{N}} \left(a_1^\dagger(k) \right)^{N_1(k)} \cdots \left(a_8^\dagger(k) \right)^{N_8(k)} |0\rangle , \tag{3.1}$$

where $a_i^\dagger(k)$ (k is a natural number) are oscillators creation operators, $N_i(k)$ are the corresponding occupation numbers, and $|0\rangle$ stands for the oscillators ground state.

Using (2.5) and (2.9) one can calculate the action of the group $G^{\rm mon}$ on the wave function (3.1). An element $Z_{\mathbf{n}} \in G^{\rm mon}$ acts on (3.1) by multiplication by the exponential factor

$$\exp\left(2\pi i \sum_{s=1}^{d} n_s \left(-\tilde{p}^s + \tilde{\theta}^{sd} \sum_{k\in\mathbf{N}} \sum_{l=1}^{8} N_l(k) k\right)\right).$$

Hence, the invariance of state vectors under the gauge transformations generated by the group $G^{\rm mon}$ leads to the quantization condition of zero modes $\tilde{p}^s$:

$$\tilde{p}^s = \tilde{n}^s + \tilde{\theta}^{sd} \sum_{k\in\mathbf{N}} \sum_{l=1}^{8} N_l(k) k \,, \tag{3.2}$$

where $\tilde{n}^s$ are integers. Quantization of the fermionic part (2.12) is straightforward. Note that the zero modes $\psi(0)$ are not dynamical. They only influence the degeneracy of states.

The energy spectrum of the system reads

$$E_{\tilde{n}_s,K} = \frac{g_{YM}^2}{2V}(\tilde{n}^s + \tilde{\theta}^{sd}K) g_{sr} (\tilde{n}^r + \tilde{\theta}^{rd}K) + \frac{V}{4g_{YM}^2}\phi_{\mu\nu}\phi^{\mu\nu} + \|K\| \,, \tag{3.3}$$

where

$$K = \sum_{k\in\mathbf{N}} \sum_{l=1}^{8} N_l(k) k + \sum_{k\in\mathbf{N}} \sum_{i=1}^{8} \mathcal{V}_i(k) k \,. \tag{3.4}$$

In the last formula $\mathcal{V}_i(k) = 0, 1$ stand for fermionic occupation numbers. For fixed numbers $\tilde{n}^s$ and K the degeneracy of the energy eigenvalue (3.3) is defined by the number of representations of K in the form (3.4) (the number of partitions). More explicitly, the degeneracy of the eigenvalue $E_{\tilde{n}_i,K}$ is given by the coefficient at the K-th power of x in the partition function

$$Z(x) = 2^8 \prod_n \left(\frac{1+x^n}{1-x^n}\right)^8 \,. \tag{3.5}$$

From general arguments (see [10, 7]) we know that the eigenvalues of operators $p^i = \mathrm{Tr}\, P^i$ and $\mathcal{P}_i = \mathrm{Tr}\, F_{ij} P^j$ obey the following quantization conditions

$$\begin{aligned} p^i &= n^i + \theta^{ij} m_j \,, \\ \mathcal{P}_j &= 2\pi m_j \,, \end{aligned}$$

where n^i, m_j ($i, j = 1, \ldots, d$) are integers. These eigenvalues are defined for a given module E and connection ∇_α on it. In transition from E to a module $\tilde{E}$ over Morita equivalent torus $T_{\tilde{\theta}}$ connections and endomorphisms on E are mapped to connections and endomorphisms on $\tilde{E}$. Thus, the integers n^i, m_j defining the eigenvalues of operators p^i and $\mathcal{P}_i$ on E transform to new integers $\tilde{n}^i$ and $\tilde{m}_j$ related to $\tilde{E}$. One can prove (see [10, 6, 7]) that the set of numbers $(-n^i, m_j)$ transform according to a vector representation of the group $\mathrm{SO}(d,d|\mathbb{Z})$. For the case at hand when E is some basic module over T_θ and $\tilde{E}$ is a free module over $T_{\tilde{\theta}} = \mathrm{End}_{T_\theta} E$ the eigenvalues $\tilde{n}^i$ are given in (3.2), and for $\tilde{\mathcal{P}}_j$ a straightforward calculation yields

$$\tilde{\mathcal{P}}_j = 2\pi \tilde{m}_j\,, \qquad \tilde{m}_j = \delta_{jd} K\,, \tag{3.6}$$

where K is given by (3.4). The integers $(\tilde{n}^i, \tilde{m}_j)$ related to the free module $\tilde{E}$ can be expressed via the integers n^i and m_j corresponding to E as follows

$$\tilde{n}^i = M^i_j n^j - N^{ij} m_j\,, \tag{3.7}$$

$$\tilde{m}_i = S^j_i m_j - R_{ij} n^j\,. \tag{3.8}$$

Conversely, the values of n^i and m_j are related to $(\tilde{n}^i, \tilde{m}_j)$ by means of the inverse matrix to (2.4):

$$g^{-1} = \begin{pmatrix} S^t & N^t \\ R^t & M^t \end{pmatrix}.$$

In transition from $\tilde{E}$ to E one should also take into account the change of metric tensors

$$\tilde{g} = A g A^t\,, \qquad A = R\theta + S\,,$$

transformation of the background field $\phi_{\mu\nu}$

$$\tilde{\phi} = A\phi A^t + \pi R A^t\,,$$

and change of the coupling constant

$$\tilde{g}_{YM} = |detA|^{1/2} g^2_{YM}$$

(see [11, 12, 6], [7] for the details). The energy of BPS states can now be written in terms of the topological numbers specified by the matrix (2.4) and quantum numbers n^i, m_j as follows

$$\begin{aligned} E = {} & \frac{(g_{YM})^2}{2V \dim E}(n^i + \theta^{ik} m_k) g_{ij} (n^j + \theta^{jl} m_l) + \\ & + \frac{V \dim E}{4 g^2_{YM}} (\pi (A^{-1}R)_{ij} + \phi_{ij}) g^{ik} g^{jl} (\pi (A^{-1}R)_{kl} + \phi_{kl}) + \\ & + \pi \sqrt{v_i g^{ij} v_j} \end{aligned} \tag{3.9}$$

where $v_i = A^{-1}_{ij}(S^k_j m_k - R_{jk} n^k)$, $A = R\theta + S$ and R and S are blocks of the matrix (2.4). Note that the expression (3.9) does not refer to a particular choice of basis in the lattice $\mathbb{Z}^d$ which was used earlier for the sake of convenience. We skipped many technical details here as the calculation essentially parallels the one made in [6] for the case $d = 2$. We also omitted the possible topological terms in formulas (3.3) and (3.9). One can easily restore them. For the cases $d = 2, 3$ one can rewrite the expression (3.9) for BPS energy spectrum in terms of the topological numbers (p, q_{ij}) (formulas (1.3) and (1.5) in the Introduction to this paper). The answer coincides with the one given in [7] and agrees with [9, 10].

Using (3.5), (3.6), and (3.8) one can also express the degeneracy of BPS states in terms of the numbers related to E. It is easy to see that the expression $\gcd(x_i)$ where $x = (x_1, \ldots, x_d) \in \mathbb{Z}^d$ is invariant under $\mathrm{SL}(d, \mathbb{Z})$ transformations. From (3.6) we know the answer for the case when the vector $\tilde{m}_i$ is directed along the d-th axis. Using an $\mathrm{SL}(d, \mathbb{Z})$ transformation and invariance of $\gcd(\tilde{m}_i)$ we obtain that in the general case the degeneracy of BPS states is specified by the number of partitions of the integer

$$K = \gcd(S_i^j m_j - R_{ij} n^j)\,.$$

For the cases $d = 2$ and $d = 3$ it is easy to find an explicit formula for a matrix (2.4) in terms of the topological numbers (p, q_{ij}). For the two-dimensional case one can take

$$g = \begin{pmatrix} p' \cdot \mathbf{1} & q'_{ij} \\ q_{ij} & p \cdot \mathbf{1} \end{pmatrix}$$

where $\mathbf{1}$ is the 2×2 identity matrix, q'_{ij} is an antisymmetric 2×2 matrix with integer entries and p' is an integer such that $q_{12} q'_{12} + p p' = 1$. Therefore, for $d = 2$ the degeneracy of 1/4 BPS states is determined by $K = \gcd(p m_i - q_{ij} n^j)$.

For $d = 3$ one can derive the following expression for $\tilde{m}_i$ in terms of the topological numbers p, q_{ij} and integers m_i, n^j

$$\tilde{m}_i = S_i^j m_j - R_{ij} n^j = \left(p \delta_j^i + (1 - p) \bar{q}_i q^j\right) m_j - q_{ij} n^j$$

where $q^j = \frac{1}{2} \epsilon^{jkl} q_{kl} (\gcd(q_{ij}))^{-1}$ and the integers $\bar{q}_i$ satisfy $\sum_{i=1}^3 q^i \bar{q}_i = 1$. For $K = \gcd(\tilde{m}_i)$ one can write a more compact formula $K = \gcd(p m_i - q_{ij} n^j, m_i \frac{1}{2} \epsilon^{ijk} q_{jk})$. Here we used the fact that the numbers (p, q_{ij}) are relatively prime. Using the same fact one can verify that

$$K = \gcd\left(p m_i - q_{ij} n^j, m_i \frac{1}{2} \epsilon^{ijk} q_{jk}\right) = \gcd\left(p m_i - q_{ij} n^j, m_i \frac{1}{2} \epsilon^{ijk} q_{jk}, m_i n^i\right).$$

The last expression agrees with the formula given in [13] for degeneracies of 1/4-BPS states on a three-dimensional commutative torus.

Let us discuss now how one can generalize our considerations to the case of arbitrary modules admitting a constant curvature connection. In this case one can reduce the problem to the consideration of BPS states in a free module having rank $N > 1$. Again, we can describe 1/4-BPS fields and reduce our problem to the quantization of an analogue of the orbifold chiral sigma model on $(\mathbb{R}^6 \times T^2)^N / S_N$. More precisely, the general formula for 1/4-BPS fields in a free module of rank $N > 1$

can be written in the same way as in the case $N = 1$ (formula (2.8)). The only difference is that in the general case $A_i(k)$, $X_I(k)$ and the zero modes $q_s \equiv A_s(0)$, $\tilde{p}^s$ are $N \times N$ matrices satisfying the conditions

$$
\begin{aligned}
\sum_{k+k'=n} [A_i(k), A_j(k')] &= 0\,, \\
\sum_{k+k'=n} [A_i(k), X_I(k')] &= 0\,, \\
\sum_{k+k'=n} [X_I(k), X_J(k')] &= 0
\end{aligned}
\tag{3.10}
$$

for any $n \in \mathbb{Z}$ and any $i, j = 1, \ldots, d$, $I, J = d+1, \ldots 9$. Introducing generating functions

$$
A_i(\phi) = \sum_{k \in \mathbb{Z}} A_i(k) e^{ik\phi} \qquad X_I(\phi) = \sum_{k \in \mathbb{Z}} X_I(k) e^{ik\phi}\,.
$$

we can express these conditions as commutation relations

$$
[A_i(\phi), A_j(\phi)] = 0\,, \qquad [A_i(\phi), X_I(\phi)] = 0\,, \qquad [X_I(\phi), X_J(\phi)] = 0\,. \tag{3.11}
$$

One can use the remaining gauge invariance to simplify further the study of BPS states. Using the commutation relations (3.11) one can prove that there exists such a matrix-valued function $u(\phi)$ satisfying the conditions that the matrices $u^{-1}(\phi)A_i(\phi)u(\phi)$ and $u^{-1}(\phi)X_I(\phi)u(\phi)$ are diagonal and $u(0) = 1$. In the generic case (more precisely in the case when for every ϕ at least one of the matrices $A_i(\phi)$, $X_I(\phi)$ has distinct eigenvalues) there exists a unique continuous function $u(\phi)$ obeying these conditions and $u(2\pi)$ is a permutation matrix. Let us stress that diagonalized matrix-valued functions $A_i(\phi)$, $X_I(\phi)$ are not periodic in general.

The above diagonalization permits us to reduce the study of 1/4-BPS states in an arbitrary free module to the study of an orbifold sigma model with the target space $(\mathbb{R}^{9-d} \times T^{d-1})^N / S_N$. A model of this kind was analyzed in [15, 16]. It was shown there that when N tends to infinity such a model is related to string theory in the light cone gauge. As it was emphasized at the end of section 2 our sigma-model is not completely standard, however the considerations of the papers we mentioned can be applied to our situation. We can conclude that in the situation when BPS-fields are dominant the SYM theory on noncommutative torus is related to string theory/M-theory. Looking more closely at this relation we find an agreement with the physical interpretation of compactifications on noncommutative tori in terms of toroidal compactifications of M-theory with non-vanishing expectation value of the three-form (see [3]). In particular, the relation between M-theory/String spectra and SYM spectra (for example see [17]) agrees with our formulas.

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